Definition Let k_1, k_2, $\ldots$, k_n be a random sample of size n from the discrete pdf $p_X(k;\theta)$, where θ is an unknown parameter. The **likelihood function**, $L(\theta)$, is the product of the pdf evaluated at the n k_i's. That is,

$$L(\theta) = \prod_{i=1}^{n} p_X(k_i;\theta)$$

Let y_1, y_2, $\ldots$, y_n be a random sample of size n from the continuous pdf $f_Y(y_i;\theta)$, where θ is an unknown parameter, the likelihood function is written

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta)$$

Note: Joint pdfs and likelihood functions look the same, but the two are actually different. Joint pdfs are multivariate functions whereas likelihood function L is a function of θ. The likelihood function L should not be considered as a function of either the k_i's or the y_i's.

Definition: Let $L(\theta) = \prod_{i=1}^{n} p_X(k_i;\theta)$ and $L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta)$ be the likelihood functions corresponding to random samples $k_1, k_2, \ldots, k_n$ and $y_1, y_2, \ldots, y_n$ drawn from the discrete pdf $p_X(k;\theta)$ and continuous pdf $f_Y(y;\theta)$, respectively, where θ is an unknown parameter. In each case, let θ_e be a value of the parameter such that $L(\theta_e) \geq L(\theta)$ for all possible values of *theta*. Then θ_e is called a *maximum likelihood estimate* for θ.

Note: It is important to understand the difference between a maximum likelihood estimate and a maximum likelihood estimator (MLE). The first is a number or an expression representing a number; the second is a random variable or a function.

(1) Imagine being handed a coin whose probability, p, of coming up heads is unknown. Your assignment is to toss the coin three times and use the resulting sequence of H's and T's to suggest a value for p. Suppose the sequence of three tosses turns out to be HHT. Based on those outcomes, what can be reasonably inferred about p?

(2) Suppose we toss a coin n times and record a set of outcomes $X_1 = k_1$, $X_2 = k_2$, $\ldots$, $X_n = k_n$. Find a maximum likelihood estimate for p.

(3) Suppose that $X_1 = 3$, $X_2 = 5$, $X_3 = 4$, and $X_4 = 2$ is a set of four independent observations representing the Poisson probability model. Find the maximum likelihood for λ.

(4) Suppose that $X_1 = k_1$, $X_2 = k_2$, $\ldots$,$X_n = k_n$ is a set of four independent observations representing the Poisson probability model. Find formula for the maximum likelihood estimate for λ.

(5) Suppose an isolated weather-reporting station has an electronic device whose time to failure is given by the exponential model

$$f_Y(y;\theta) = \frac{1}{\theta}\, e^{-y/\theta}, \quad 0 \leq y < \infty;\ 0 < \theta < \infty$$

The station also has a spare device, so the time until this instrument is not available is the sum of these two exponential pdfs, which is

$$f_Y(y;\theta) = \frac{1}{\theta^2}\, y\, e^{-y/\theta}, \quad 0 \leq y < \infty;\ 0 < \theta < \infty$$

Five data points have been collected—9.2, 5.6, 18.4, 12.1, and 10.7. Find the maximum likelihood estimate for θ.

(6) Suppose $y_1, y_2, \ldots, y_n$ is a set of measurements representing an exponential pdf with $\lambda = 1$ but with an unknown "threshold" parameter, θ. That is,

$$f_Y(y;\theta) = e^{-(y-\theta)},\ y \geq \theta;\ \ \theta > 0.$$

Find the maximum likelihood estimate for θ.

(7) Let Y_1,Y_2, $\ldots$, Y_n be a random sample from a normal distribution with mean μ and variance σ^2. Find the maximum likelihood estimators of μ and σ^2.

(8) Let Y_1,Y_2, $\ldots$, Y_n be a random sample observations from a uniform distribution with probability density function $f(y_i|\theta) = \dfrac{1}{\theta}$, for $0 \leq y_i \leq \theta$ and $i = 1, 2, \ldots, n$. Find the MLE of θ.

(9) In Problem 2, we found that the MLE of the binomial proportion p is given by $\hat{p} = \dfrac{X}{n} = \dfrac{1}{n}\sum_{i=1}^{n} k_i$: the fraction of successes in the total number of trials n. What is the MLE for the variance of X?